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# REGULAR SUBANALYTIC COVERS

ADAM PARUSIŃSKI

**ABSTRACT.** Let  $U$  be an open relatively compact subanalytic subset of a real analytic manifold  $M$ . We show that there exists a “finite linear covering” (in the sense of Guillermou-Schapira) of  $U$  by subanalytic open subsets of  $U$  homeomorphic to an open ball.

We also show that the characteristic function of  $U$  can be written as a finite linear combination of characteristic functions of open relatively compact subanalytic subsets of  $M$  homeomorphic, by subanalytic and bi-lipschitz maps, to an open ball.

Let  $M$  be a real analytic manifold of dimension  $n$ . In this paper we study the subalgebra  $\mathcal{S}(M)$  of integer valued functions on  $M$  generated by characteristic functions of relatively compact open subanalytic subsets of  $M$  (or equivalently by characteristic functions of compact subanalytic subsets of  $M$ ). As we show this algebra is generated by characteristic functions of open subanalytic sets with Lipschitz regular boundaries. More precisely, we call a relatively compact open subanalytic subset  $U \subset M$  an *open subanalytic Lipschitz ball* if its closure is subanalytically bi-Lipschitz homeomorphic to the unit ball of  $\mathbb{R}^n$ . Here we assume that  $M$  is equipped with a Riemannian metric. Any two such metrics are equivalent on relatively compact sets and hence the above definition is independent of the choice of a metric.

**Theorem 0.1.** *The algebra  $\mathcal{S}(M)$  is generated by characteristic functions of open subanalytic Lipschitz balls.*

That is to say if  $U$  is a relatively compact open subanalytic subset of  $M$  then its characteristic function  $1_U$  is a finite integral linear combination of characteristic functions  $1_{W_1}, \dots, 1_{W_m}$ , where the  $W_j$  are open subanalytic Lipschitz balls. Note that, in general,  $U$  cannot be covered by finitely many subanalytic Lipschitz balls, as it is easy to see for  $\{(x, y) \in \mathbb{R}^2; y^2 < x^3, x < 1\}$ ,  $M = \mathbb{R}^2$ , due to the presence of cusps. Nevertheless we show the existence of a “regular” cover in the sense that we control the distance to the boundary.

**Theorem 0.2.** *Let  $U$  be an open relatively compact subanalytic subset of  $M$ . Then there exist a finite cover  $U = \bigcup_i U_i$  by open subanalytic sets such that :*

- (1) *every  $U_i$  is subanalytically homeomorphic to an open  $n$ -dimensional ball;*
- (2) *there is  $C > 0$  such that for every  $x \in U$ ,  $\text{dist}(x, M \setminus U) \leq C \max_i \text{dist}(x, M \setminus U_i)$*

The proof of Theorem 0.1 is based on the classical cylindrical decomposition and the L-regular decomposition of subanalytic sets, cf. [4], [9], [10]. L-regular sets are natural multidimensional generalization of classical cusps. We recall them briefly in Subsection 1.6.

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For the proof of Theorem 0.2 we need also the regular projection theorem, cf. [7], [8], [9], that we recall in Subsection 1.4.

We also show the following strengthening of Theorem 0.2.

**Theorem 0.3.** *In Theorem 0.2 we may require additionally that all  $U_i$  are open  $L$ -regular cells.*

For an open  $U \subset M$  we denote  $\partial U = \overline{U} \setminus U$ .

## 1. PROOFS

**1.1. Reduction to the case  $M = \mathbb{R}^n$ .** Let  $U$  be an open relatively compact subanalytic subset of  $M$ . Choose a finite cover  $\overline{U} \subset \bigcup_i V_i$  by open relatively compact sets such that for each  $V_i$  there is an open neighborhood of  $\overline{V_i}$  analytically diffeomorphic to  $\mathbb{R}^n$ . Then there are finitely many open subanalytic  $U_{ij}$  such that  $U_{ij} \subset V_i$  and  $1_U$  is a combination of  $1_{U_{ij}}$ . Thus it suffices to show Theorem 0.1 for relatively compact open subanalytic subsets of  $\mathbb{R}^n$ .

Similarly, it suffices to show Theorems 0.2 and 0.3 for  $M = \mathbb{R}^n$ . Indeed, it follow from the observation that the function

$$x \rightarrow \max_i \text{dist}(x, M \setminus V_i)$$

is continuous and nowhere zero on  $\bigcup_i V_i$  and hence bounded from below by a nonzero constant  $c > 0$  on  $\overline{U}$ . Then

$$\text{dist}(x, M \setminus U) \leq C_1 \leq c^{-1} C_1 \max_i \text{dist}(x, M \setminus V_i)$$

where  $C_1$  is the diameter of  $\overline{U}$  and hence, if  $c^{-1} C_1 \geq 1$ ,

$$\text{dist}(x, M \setminus U) \leq c^{-1} C_1 \max_i (\min\{\text{dist}(x, M \setminus U), \text{dist}(x, M \setminus V_i)\}).$$

Now if for each  $U \cap V_i$  we choose a cover  $\bigcup_j U_{ij}$  satisfying the statement of Theorem 0.2 or 0.3 then for  $x \in U$

$$\begin{aligned} \text{dist}(x, M \setminus U) &\leq c^{-1} C_1 \max_i (\min\{\text{dist}(x, M \setminus U), \text{dist}(x, M \setminus V_i)\}) \\ &\leq c^{-1} C_1 \max_i \text{dist}(x, M \setminus U \cap V_i) \leq C c^{-1} C_1 \max_{ij} \text{dist}(x, M \setminus U_{ij}) \end{aligned}$$

Thus the cover  $\bigcup_{i,j} U_{ij}$  satisfies the claim of Theorem 0.2, resp. of Theorem 0.3.

**1.2. Regular projections.** We recall after [8], [9] the subanalytic version of the regular projection theorem of T. Mostowski introduced originally in [7] for complex analytic sets germs.

Let  $X \subset \mathbb{R}^n$  be subanalytic. For  $\xi \in \mathbb{R}^{n-1}$  we denote by  $\pi_\xi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  the linear projection parallel to  $(\xi, 1) \in \mathbb{R}^{n-1} \times \mathbb{R}$ . Fix constants  $C, \varepsilon > 0$ . We say that  $\pi = \pi_\xi$  is  $(C, \varepsilon)$ -regular at  $x_0 \in \mathbb{R}^n$  (with respect to  $X$ ) if

- (a)  $\pi|_X$  is finite;

(b) the intersection of  $X$  with the open cone

$$(1.1) \quad \mathcal{C}_\varepsilon(x_0, \xi) = \{x_0 + \lambda(\eta, 1); |\eta - \xi| < \varepsilon, \lambda \in \mathbb{R} \setminus 0\}$$

is empty or a finite disjoint union of sets of the form

$$\{x_0 + \lambda_i(\eta)(\eta, 1); |\eta - \xi| < \varepsilon\},$$

where  $\lambda_i$  are real analytic nowhere vanishing functions defined on  $|\eta - \xi| < \varepsilon$ .

(c) the functions  $\lambda_i$  from (b) satisfy for all  $|\eta - \xi| < \varepsilon$

$$\|\text{grad } \lambda_i(\eta)\| \leq C|\lambda_i(\eta)|,$$

We say that  $\mathcal{P} \subset \mathbb{R}^{n-1}$  defines a set of regular projections for  $X$  if there exists  $C, \varepsilon > 0$  such that for every  $x_0 \in \mathbb{R}^n$  there is  $\xi \in \mathcal{P}$  such that  $\pi_\xi$  is  $(C, \varepsilon)$ -regular at  $x_0$ .

**Theorem 1.1.** *[8], [9] Let  $X$  be a compact subanalytic subset of  $\mathbb{R}^n$  such that  $\dim X < n$ . Then the generic set of  $n+1$  vectors  $\xi_1, \dots, \xi_{n+1}$ ,  $\xi_i \in \mathbb{R}^{n-1}$ , defines a set of regular projections for  $X$ .*

Here by generic we mean in the complement of a subanalytic nowhere dense subset of  $(\mathbb{R}^{n-1})^{n+1}$ .

**1.3. Cylindrical decomposition.** We recall the first step of a basic construction called the cylindrical algebraic decomposition in semialgebraic geometry or the cell decomposition in o-minimal geometry, for details see for instance [2], [3].

Set  $X = \bar{U} \setminus U$ . Then  $X$  is a compact subanalytic subset of  $\mathbb{R}^n$  of dimension  $n - 1$ . We denote by  $Z \subset X$  the set of singular points of  $X$  that is the complement in  $X$  of the set

$$\text{Reg}(X) := \{x \in X; (X, x) \text{ is the germ of a real analytic submanifold of dimension } n - 1\}.$$

Then  $Z$  is closed in  $X$ , subanalytic and  $\dim Z \leq n - 2$ .

Assume that the standard projection  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  restricted to  $X$  is finite. Denote by  $\Delta_\pi \subset \mathbb{R}^{n-1}$  the union of  $\pi(Z)$  and the set of critical values of  $\pi|_{\text{Reg}(X)}$ . Then  $\Delta_\pi$ , called the discriminant set of  $\pi$ , is compact and subanalytic. It is clear that  $\pi(\bar{U}) = \pi(U) \cup \Delta_\pi$ .

**Proposition 1.2.** *Let  $U' \subset \pi(U) \setminus \Delta_\pi$  be open and connected. Then there are finitely many bounded real analytic functions  $\varphi_1 < \varphi_2 < \dots < \varphi_k$  defined on  $U'$ , such that  $X \cap \pi^{-1}(U')$  is the union of graphs of  $\varphi_i$ 's. In particular,  $U \cap \pi^{-1}(U')$  is the union of the sets*

$$\{(x', x_n) \in \mathbb{R}^n; x' \in U', \varphi_i(x') < x_n < \varphi_{i+1}(x')\},$$

and moreover, if  $U'$  is subanalytically homeomorphic to an open  $(n - 1)$ -dimensional ball, then each of these sets is subanalytically homeomorphic to an open  $n$ -dimensional ball.

**1.4. The case of a regular projection.** Fix  $x_0 \in U$  and suppose that  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  is  $(C, \varepsilon)$ -regular at  $x_0 \in \mathbb{R}^n$  with respect to  $X$ . Then the cone (1.1) contains no point of  $Z$ . By [9] Lemma 5.2, this cone contains no critical point of  $\pi|_{\text{Reg}(X)}$ , provided  $\varepsilon$  is chosen sufficiently small (for fixed  $C$ ). In particular,  $x'_0 = \pi(x_0) \notin \Delta_\pi$ .

In what follows we fix  $C, \varepsilon > 0$  and suppose  $\varepsilon$  small. We denote the cone (1.1) by  $\mathcal{C}$  for short. Then for  $\tilde{C}$  sufficiently large, that depends only on  $C$  and  $\varepsilon$ , we have

$$(1.2) \quad \text{dist}(x_0, X \setminus \mathcal{C}) \leq \tilde{C} \text{dist}(x'_0, \pi(X \setminus \mathcal{C})) \leq \tilde{C} \text{dist}(x'_0, \Delta_\pi).$$

The first inequality is obvious, the second follow from the fact that the singular part of  $X$  and the critical points of  $\pi|_{\text{Reg}(X)}$  are both outside the cone.

**1.5. Proof of Theorem 0.2.** Induction on  $n$ . Set  $X = \bar{U} \setminus U$  and let  $\pi_{\xi_1}, \dots, \pi_{\xi_{n+1}}$  be a set of  $(C, \varepsilon)$ -regular projections with respect to  $X$ . To each of these projections we apply the cylindrical decomposition. More precisely, let us fix one of these projections that for simplicity we suppose standard and denote it by  $\pi$ . Then we apply the inductive assumption to  $\pi(U) \setminus \Delta_\pi$ . Thus let  $\pi(U) \setminus \Delta_\pi = \bigcup U'_i$  be a finite cover satisfying the statement of Theorem 0.2. Applying to each  $U'_i$  Proposition 1.2 we obtain a family of cylinders that covers  $U \setminus \pi^{-1}(\Delta_\pi)$ . In particular they cover the set of those points of  $U$  at which  $\pi$  is  $(C, \varepsilon)$ -regular.

**Lemma 1.3.** *Suppose  $\pi$  is  $(C, \varepsilon)$ -regular at  $x_0 \in U$ . Let  $U'$  be an open subanalytic subset of  $\pi(U) \setminus \Delta_\pi$  such that  $x'_0 = \pi(x_0) \in U'$  and*

$$(1.3) \quad \text{dist}(x'_0, \Delta_\pi) \leq \tilde{C} \text{dist}(x'_0, \partial U'),$$

*with  $\tilde{C} \geq 1$  for which (1.2) holds. Then*

$$(1.4) \quad \text{dist}(x_0, X) \leq (\tilde{C})^2 \text{dist}(x_0, \partial U_1),$$

*where  $U_1$  is the member of cylindrical decomposition of  $U \cap \pi^{-1}(U')$  containing  $x_0$ .*

*Proof.* We decompose  $\partial U_1$  into two parts. The first one is vertical, i.e. contained in  $\pi^{-1}(\partial U')$ , and the second part is contained in  $X$ . If  $\text{dist}(x_0, \partial U_1) < \text{dist}(x_0, X)$  then the distance to the vertical part realizes the distance of  $x_0$  to  $\partial U_1$  and  $\text{dist}(x_0, \partial U_1) = \text{dist}(x'_0, \partial U')$ . Hence

$$(1.5) \quad \text{dist}(x_0, \partial U_1) = \min\{\text{dist}(x_0, X), \text{dist}(x'_0, \partial U')\}.$$

If  $\text{dist}(x_0, \partial U_1) = \text{dist}(x_0, X)$  then (1.4) holds with  $\tilde{C} = 1$ , otherwise by (1.2) and (1.3)

$$(1.6) \quad \text{dist}(x_0, X \setminus \mathcal{C}) \leq \tilde{C} \text{dist}(x'_0, \Delta_\pi) \leq (\tilde{C})^2 \text{dist}(x'_0, \partial U') = (\tilde{C})^2 \text{dist}(x_0, \partial U_1).$$

□

Thus to complete the proof of Theorem 0.2 it suffices to show that (1.3) holds if  $U'$  is an element of the cover  $\pi(U) \setminus \Delta_\pi = \bigcup U'_i$  for which  $\text{dist}(x'_0, \partial \pi(U)) \leq \tilde{C} \text{dist}(x'_0, \partial U')$ . This follows from the inclusion  $\partial \pi(U) \subset \Delta_\pi$  that gives  $\text{dist}(x'_0, \Delta_\pi) \leq \text{dist}(x'_0, \partial \pi(U))$ . This ends the proof of Theorem 0.2.

**1.6. L-regular sets.** Let  $Y \subset \mathbb{R}^n$  be subanalytic,  $\dim Y = n$ . Then  $Y$  is called *L-regular* (with respect to given system of coordinates) if

- (1) if  $n = 1$  then  $Y$  is a non-empty closed bounded interval;
- (2) if  $n > 1$  then  $Y$  is of the form

$$(1.7) \quad Y = \{(x', x_n) \in \mathbb{R}^n; f(x') \leq x_n \leq g(x'), x' \in Y'\},$$

where  $Y' \subset \mathbb{R}^{n-1}$  is L-regular,  $f$  and  $g$  are continuous subanalytic functions defined in  $Y'$ . It is also assumed that on the interior of  $Y'$ ,  $f$  and  $g$  are analytic, satisfy  $f < g$ , and have the first order partial derivatives bounded.

If  $\dim Y = k < n$  then we say that  $Y$  is *L-regular* (with respect to given system of coordinates) if

$$(1.8) \quad Y = \{(y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}; z = h(y), y \in Y'\},$$

where  $Y' \subset \mathbb{R}^k$  is L-regular,  $\dim Y' = k$ ,  $h$  is a continuous subanalytic map defined on  $Y'$ , such that  $h$  is real analytic on the interior of  $Y'$ , and has the first order partial derivatives bounded.

We say that  $Y$  is *L-regular* if it is L-regular with respect to a linear (or equivalently orthogonal) system of coordinates on  $\mathbb{R}^n$ .

We say that  $A \subset \mathbb{R}^n$  is an *L-regular cell* if  $A$  is the relative interior of an L-regular set. That is, it is the interior of  $A$  if  $\dim A = n$ , and it is the graph of  $h$  restricted to  $\text{Int}(Y')$  for an L-regular set of the form (1.8). By convention, every point is a zero-dimensional L-regular cell.

By [4], see also Lemma 2.2 of [9] and Lemma 1.1 of [5], L-regular sets and L-regular cells satisfy the following property, called in [4] quasi-convexity. We say that  $Z \subset \mathbb{R}^n$  is *quasi-convex* if there is a constant  $C > 0$  such that every two points  $x, y$  of  $Z$  can be connected in  $Z$  by a continuous subanalytic arc of length bounded by  $C\|x - y\|$ . It can be shown that for an L-regular set or cell  $Y$  of the form (1.7) or (1.8) the constant  $C$  depends only on  $n$ , the analogous constant for  $Y'$ , and the bounds on first order partial derivatives of  $f$  and  $g$ , resp.  $h$ . By Lemma 2.2 of [9], an L-regular cell is subanalytically homeomorphic to the (open) unit ball.

Let  $Y$  be a subanalytic subset of a real analytic manifold  $M$ . We say that  $Y$  is *L-regular* if there exists its neighborhood  $V$  in  $M$  and an analytic diffeomorphism  $\varphi : V \rightarrow \mathbb{R}^n$  such that  $\varphi(Y)$  is L-regular. Similarly we define an *L-regular cell* in  $M$ .

**1.7. Proof of Theorem 0.3.** Fix a constant  $C_1$  sufficiently large and a projection  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  that is assumed, for simplicity, to be the standard one. We suppose that  $\pi$  restricted to  $X = \partial U$  is finite. We say that  $x' \in \pi(U) \setminus \Delta_\pi$  is  *$C_1$ -regularly covered* if there is a neighborhood  $\tilde{U}'$  of  $x'$  in  $\pi(U) \setminus \Delta_\pi$  such that  $X \cap \pi^{-1}(\tilde{U}')$  is the union of graphs of analytic functions with the first order partial derivatives bounded (in the absolute value) by  $C_1$ . Denote by  $U'(C_1)$  the set of all  $x' \in \pi(U) \setminus \Delta_\pi$  that are  $C_1$  regularly covered. Then  $U'(C_1)$  is open (if we use strict inequalities while defining it) and subanalytic. By Lemma 5.2 of [9], if  $\pi$  is a  $(C, \varepsilon)$ -regular projection at  $x_0$  then  $x'_0$  is  $C_1$ -regularly covered, for  $C_1$  sufficiently big  $C_1 \geq C_1(C, \varepsilon)$ . Moreover we have the following result.

**Lemma 1.4.** *Given positive constants  $C, \varepsilon$ . Suppose that the constants  $\tilde{C}$  and  $C_1$  are chosen sufficiently big,  $C_1 \geq C_1(C, \varepsilon)$ ,  $\tilde{C} \geq \tilde{C}(C, \varepsilon)$ . Let  $\pi$  be  $(C, \varepsilon)$ -regular at  $x_0 \notin X$  and let*

$$V' = \{x' \in \mathbb{R}^{n-1}; \text{dist}(x', x'_0) < (\tilde{C})^{-1} \text{dist}(x_0, X \cap \mathcal{C})\}.$$

*Then  $\pi^{-1}(V') \cap X \cap \mathcal{C}$  is the union of graphs of  $\varphi_i$  with all first order partial derivatives bounded (in the absolute value) by  $C_1$ . Moreover, then either  $\pi^{-1}(V') \cap (X \setminus \mathcal{C}) = \emptyset$  or*

$$\text{dist}(x'_0, \Delta_\pi) = \text{dist}(x'_0, \pi(X \setminus \mathcal{C})) \leq \text{dist}(x'_0, \partial U'(C_1)).$$

*Proof.* We only prove the second part of the statement since the first part follows from Lemma 5.2 of [9]. If  $\pi^{-1}(V') \cap X \setminus \mathcal{C} \neq \emptyset$  then any point of  $\pi(X \setminus \mathcal{C})$  realizing  $\text{dist}(x'_0, \pi(X \setminus \mathcal{C}))$  must be in the discriminant set  $\Delta_\pi$ .  $\square$

We now apply to  $U'(C_1)$  the inductive hypothesis and thus assume that  $U'(C_1) = \bigcup U'_i$  is a finite regular cover by open  $L$ -regular cells. Fix one of them  $U'$  and let  $U_1$  be a member of the cylindrical decomposition of  $U \cap \pi^{-1}(U')$ . Then  $U_1$  is an  $L$ -regular cell. Let  $x_0 \in U_1$ . We apply to  $x_0$  Lemma 1.4.

If  $\pi^{-1}(V') \cap (X \setminus \mathcal{C}) = \emptyset$  then

$$\text{dist}(x_0, X) \leq \text{dist}(x_0, X \cap \mathcal{C}) \leq \tilde{C} \text{dist}(x'_0, \partial U'(C_1)) \leq \tilde{C}^2 \text{dist}(x'_0, \partial U'),$$

where the second inequality follows from the first part of Lemma 1.4 and the last inequality by the induction hypothesis. Then  $\text{dist}(x_0, X) \leq \tilde{C}^2 \text{dist}(x_0, \partial U_1)$  follows from (1.5).

Otherwise,  $\text{dist}(x'_0, \Delta_\pi) \leq \text{dist}(x'_0, \partial U'(C_1)) \leq \tilde{C} \text{dist}(x'_0, \partial U')$  and the claim follows from Lemma 1.3. This ends the proof.

**1.8. Proof of Theorem 0.1.** The proof is based on the following result.

**Theorem 1.5.** [Theorem A of [4]] *Let  $Z_i \subset \mathbb{R}^n$  be a finite family of bounded subanalytic sets. Then there is a finite disjoint collection  $\{A_j\}$  of  $L$ -regular cells such that each  $Z_i$  is the disjoint union of some of  $A_j$ .*

Similar results in the (more general) o-minimal set-up are proven in [5] and [10].

Let  $U$  be a relatively compact open subanalytic subset of  $\mathbb{R}^n$ . By Theorem 1.5,  $U$  is a disjoint union of  $L$ -regular cells and hence it suffices to show the statement of Theorem 0.1 for an  $L$ -regular cell. We consider first the case of an open  $L$ -regular cell. Thus suppose that

$$(1.9) \quad U = \{(x', x_n) \in \mathbb{R}^n; f(x') < x_n < g(x'), x' \in U'\},$$

where  $U'$  is a relatively compact  $L$ -regular cell,  $f$  and  $g$  are subanalytic and analytic functions on  $U'$  with the first order partial derivatives bounded. Then, by the quasi-convexity of  $U'$ ,  $f$  and  $g$  are Lipschitz. By an extension formula of [6], see also [11] and [1], we may suppose that  $f$  and  $g$  are restrictions of Lipschitz subanalytic functions, that we denote later also by  $f$  and  $g$ , defined everywhere on  $\mathbb{R}^{n-1}$  and satisfying  $f \leq g$ . Indeed, this extension of  $f$  is given by

$$\tilde{f}(p) = \sup_{q \in U'} f(q) - L\|p - q\|,$$

where  $L$  is the Lipschitz constant of  $f$ . Then  $\tilde{f}$  is Lipschitz with the same constant as  $f$  and subanalytic. Therefore by the inductive assumption on dimension we may assume that  $U$  is given by (1.9) with  $U'$  a subanalytic Lipschitz ball. Denote  $U$  by  $U_{f,g}$  to stress its dependence on  $f$  and  $g$  (with  $U'$  fixed). Then

$$1_{U_{f,g}} = 1_{U_{f-1,g}} + 1_{U_{f,g+1}} - 1_{U_{f-1,g+1}}$$

and  $U_{f-1,g}$ ,  $U_{f,g+1}$  and  $U_{f-1,g+1}$  are open subanalytic Lipschitz balls.

Suppose now that

$$(1.10) \quad U = \{(y, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}; z = h(y), y \in U'\},$$

where  $U'$  is an open L-regular cell of  $\mathbb{R}^k$ ,  $h$  is a subanalytic and analytic map defined on  $U'$  with the first order partial derivatives bounded. Hence  $h$  is Lipschitz. We may again assume that  $h$  is the restriction of a Lipschitz subanalytic map  $h : \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$  and then, by the inductive hypothesis, that  $U'$  is a subanalytic Lipschitz ball. Let

$$U_\emptyset = \{(y, z) \in U' \times \mathbb{R}^{n-k}; h_i(y) - 1 < z_i < h_i(y) + 1, i = 1, \dots, n - k\}$$

For  $I \subset \{1, \dots, n - k\}$  we denote

$$U_I = \{(y, z) \in U_\emptyset; z_i \neq h_i(y) \text{ for } i \in I\}.$$

Note that each  $U_I$  is the disjoint union of  $2^{|I|}$  of open subanalytic Lipschitz balls and that

$$1_U = \sum_{I \subset \{1, \dots, n-k\}} (-1)^{|I|} 1_{U_I}.$$

This ends the proof.

## 2. REMARKS ON THE O-MINIMAL CASE

It would be interesting to know whether the main theorems of this paper, Theorems 0.1, 0.2, 0.3, hold true in an arbitrary o-minimal structure in the sense of [3], i.e. if we replace the word "subanalytic" by "definable in an o-minimal structure", and fix  $M = \mathbb{R}^n$ . This is the case for Proposition 1.2 and Theorem 1.5 by [3], resp. [5], [10], and therefore Theorem 0.1 holds true in the o-minimal set-up. But it is not clear whether the analog of Theorem 1.1 holds in an arbitrary o-minimal structure. Its proof in [8] uses Puiseux Theorem with parameters in an essential way. Thus we state the following questions.

**Question 2.1.** *Does the regular projections theorem, Theorem 1.1, hold true in an arbitrary o-minimal structure?*

**Question 2.2.** *Do Theorems 0.2, 0.3, hold true in an arbitrary o-minimal structure?*

One would expect the positive answers for the polynomially bounded o-minimal structures, though even this case is not entirely obvious.

## REFERENCES

- [1] S. Banach, *Wstęp do teorii funkcji rzeczywistych*, Monografie Matematyczne, Warszawa-Wrocław, 1951, (in Polish)
- [2] M. Coste, *An introduction to o-minimal geometry*, Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica, Istituti Editoriali e Poligrafici Internazionali, Pisa (2000), <http://perso.univ-rennes1.fr/michel.coste/polyens/OMIN.pdf>
- [3] L. van den Dries, *Tame Topology and O-minimal Structures*, London Math. Soc. Lecture Note **248**, Cambridge Univ. Press 1998
- [4] K. Kurdyka, *On a Subanalytic Stratification Satisfying a Whitney Property with Exponent 1*, in Real Algebraic Geometry, Proceedings, Rennes 1991, Lecture Notes in Math. **1524**, eds M. Coste et al., (1992), 316–323
- [5] K. Kurdyka, A. Parusiński, *Quasi-convex decomposition in o-minimal structures. Application to the gradient conjecture*. Singularity theory and its applications, 137177, Adv. Stud. Pure Math., 43, Math. Soc. Japan, Tokyo, 2006.
- [6] E. J. McShane, *Extension of range of functions*, Bull. Amer. Math. Soc. **40** (1934), 837–842
- [7] T. Mostowski, *Lipschitz Equisingularity*, Dissertationes Math. **243** (1985).



- [8] A. Parusiński, *Regular projection for sub-analytic sets*, C. R. Acad. Sci. Paris, Série I **307**, (1988), 343–347.
- [9] A. Parusiński, *Lipschitz stratification of subanalytic sets*, Annales Sci. Éc. Norm. Sup. **27**, 6 (1994), 661–696
- [10] W. Pawłucki, *Lipschitz cell decomposition in o-minimal structures. I*. Illinois J. Math. 52 (2008), no. 3, 1045–1763
- [11] H. Whitney, *Analytic extensions of differentiable functions defined on closed sets*. Trans. Amer. Math. Soc. 36 (1934), no. 1, 63–89

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